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Methods of thermal field theory

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Abstract : We introduce the basic ideas of thermal field theory and review its path integral formulation. We then discuss the problems of QCD theory at high and at low temperatures. At high temperature the naive perturbation expansion breaks down and is cured by resummation. We illustrate this improved perturbation expansion with the $g^2\phi^4$ theory and then sketch its application to find the gluon damping rate in QCD theory. At low temperature the hadronic phase is described systematically by the chiral perturbation theory. The results obtained from this theory for the quark and the gluon condensates are discussed.

Keywords : Thermal field theory, QCD theory, chiral perturbation expansion

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1. Introduction

Thermal field theory has grown into a vast subject. There has been a number of theoretical developments, like the resummation at high temperature, chiral perturbation theory at low temperature, non-equilibrium formalism, *etc.* It has also been applied to topics ranging from cosmology to heavy ion collisions in the laboratory. For most of the applications, however, it is difficult to formulate the problem in a way which is realistic and at the same time amenable to an easy theoretical study.

In this review we shall not deal with any of the applications in particular; instead, we shall discuss some of the theoretical developments in the QCD theory at high and at low temperatures. That is, we discuss the methods available to find the properties of the QCD medium in thermal equilibrium and of particles propagating through it in the hadronic and in the quark-gluon phase. Several well-written and much more complete reviews exist in this area [1–3].

When the temperature is low, the system consists predominantly of pions. Chiral perturbation theory is eminently suitable to evaluate all the physical properties of the system. As the temperature is increased, the interaction among the pions become strong and heavier hadronic degrees of freedom are excited. At some point a phase transition presumably takes place giving rise to the quark-gluon plasma. There is no analytic method based directly on QCD theory to discuss this transition region. When the temperature is high enough, the quark-gluon interaction becomes weak so that the usual perturbation expansion is expected to be valid. However, this naive expectation is not realised : when the external momenta are small compared to the temperature, loop contributions are of the same order as the tree level contributions. A resummation is thus needed to restore the validity of the perturbation expansion.

In Section 2 we discuss the basic ideas of thermal field theory and bring out its similarity with the conventional (zero temperature) field theory. In Section 3 the path integral formalism is obtained which gives rise to the real and the imaginary time versions. An application of the real time formalism to the thermal state in the early universe is also included here. In Section 4 we describe the resummation procedure at high temperature. In Section 5 we briefly introduce the elements of the chiral perturbation theory and discuss the results of the quark and the gluon condensates it predicts at low temperature. We conclude in Section 6.

2. Basic ideas

Conventional (zero temperature) quantum field theory describes the interaction of a few fundamental particles in the vacuum. Thermal field theory extends it to describe the interactions in a statistical system in thermal equilibrium. (We do not discuss non-equilibrium conditions in this review, except for a special type of non-equilibrium appropriate to the early universe.)

Despite apparent differences, the conventional and the thermal field theories can be developed in close parallels. This is because the Boltzmann weight factor, $e^{-\beta H}$, becomes the time evolution operator in quantum mechanics on identifying the inverse temperature β, with the imaginary time, $+it$. It is the choice of the time path which distinguishes the different formulations of the thermal theory as well as the conventional one.

The basic quantities are the thermal averages of operators. Thus for an operator A we have

$$< A > = \mathrm{Tr}(e^{-\beta H} A)/Z, \tag{2.1}$$

where the Tr (ace) is to be evaluated over a complete set of states of the system and Z is the partition function, $Z = \mathrm{Tr}\, e^{-\beta H}$. For an operator in the Heisenberg representation, we have, suppressing the space dependence,

$$e^{iHt'} A(t) e^{-iHt'} = A(t+t'),$$

so that
$$e^{\beta H} A(t) e^{-\beta H} = A(t-i\beta). \tag{2.2}$$

Consider now the thermal average of the product of two operators A and B,

$$\begin{aligned}\langle A(t)B(t')\rangle &= Z^{-1}\,\mathrm{Tr}\left(e^{-\beta H}A(t)B(t')\right)\\ &= Z^{-1}\,\mathrm{Tr}\left(e^{-\beta H}e^{\beta H}B(t')e^{-\beta H}A(t)\right)\\ &= \langle B(t'-i\beta)A(t)\rangle,\end{aligned} \tag{2.3}$$

where we have used the cyclicity of the trace in the second line and eq. (2.2) in the third line. This equation, expressing thermal equilibrium condition, is called the Kubo-Martin-Schwinger (KMS) condition.

A correlation function like (2.3) is not defined everywhere in the complex time plane. To see this, evaluate the trace over a complete set of eigenstates of the Hamiltonian, $H|E_n\rangle = E_n|E_n\rangle$ and then insert the same complete set between the operators to extract their time dependence,

$$\begin{aligned}\langle A(t)B(t')\rangle &= Z^{-1}\sum_m e^{-\beta E_m}\langle m|A(t)B(t')|m\rangle\\ &= Z^{-1}\sum_{m,n} e^{-iE_n(t-t')}e^{iE_m(t-t'+i\beta)}\langle m|A(0)|n\rangle\langle n|B(0)|m\rangle\cdot\end{aligned}$$

The finiteness of the individual terms for $E_k \to \infty$ define the strip of analyticity,

$$-\beta \le \mathrm{Im}\,(t-t') \le 0. \tag{2.4}$$

Now consider the time ordered thermal propagator for a real scalar field of mass m,

$$\langle T_c\phi(x)\phi(x')\rangle = \theta_c(t-t')\langle\phi(x)\phi(x')\rangle + \theta_c\langle\phi(x')\phi(x)\rangle \tag{2.5}$$

or,

$$D_\beta(x-x') = \theta_c(t-t')D_\beta^+(x-x') + \theta_c(t'-t)D_\beta^-(x-x'). \tag{2.6}$$

Here θ_c generalises the usual θ-function to an oriented contour. The KMS condition (2.3) applied to the thermal propagator becomes

$$D_\beta^-(t-t',\mathbf{x}-\mathbf{x}') = D_\beta^+(t-t'-i\beta,\mathbf{x}-\mathbf{x}'). \tag{2.7}$$

For imaginary times the same condition shows that the euclidean propagator can be continued outside the interval $(0,\beta)$ as a periodic function of euclidean time.

Note that the thermal propagator satisfies the same differential equation as the one at $T=0$,

$$(\Box - m^2)D_\beta(x-x') = \delta^4(x-x'). \tag{2.8}$$

It is only in the boundary condition that the thermal propagator differs from the $T = 0$ propagator. The fact that the thermal propagator satisfies the same differential equation as at $T = 0$ implies that the short distance singularities of the propagator is the same as at $T = 0$. Thus the same renormalization counterterms, needed to remove the ultraviolet divergence of the theory at zero temperature, will also suffice to make the thermal field theory divergence free.

3. Path integral formulation

Let $|\phi(\boldsymbol{x}),t\rangle$ be the basis ket in the Heisenberg picture, being the eigenstate of the field operator $\phi(\boldsymbol{x},t)$ with eigenvalue $\phi(\boldsymbol{x})$,

$$\phi(\boldsymbol{x},t)|\phi(\boldsymbol{x}),t\rangle = \phi(\boldsymbol{x})|\phi(\boldsymbol{x}),t\rangle.$$

The basis kets evolve in time as

$$|\phi(\boldsymbol{x}),t\rangle = e^{iHt}|\phi(\boldsymbol{x})\rangle.$$

The Feynman functional formula giving the transition amplitude in going from $\phi_1(\boldsymbol{x})$ at time t_1 to $\phi_2(\boldsymbol{x})$ at time t_2 is

$$\begin{aligned}\langle\phi_2(\boldsymbol{x}),t_2|\phi_1(\boldsymbol{x}),t_1\rangle &= \langle\phi_2(\boldsymbol{x})|e^{-iH(t_2-t_1)}|\phi_1(\boldsymbol{x})\rangle\\ &= N\int[d\phi]e^{i\int_{t_1}^{t_2}dt\int d^3xL(\phi,d\phi/dt)},\end{aligned}\tag{3.1}$$

where L is the Lagrangian. To arrive at the partition function, we let $i(t_2-t_1)=\beta$ and set $t_1=-T$, where T is arbitrary at the moment [4]. The trace operation requires $\int[d\phi]$ to go over all periodic paths, $\phi(-T,\boldsymbol{x})=\phi(-T-i\beta,\boldsymbol{x})$ and to integrate the resulting expression over the end values of the field. We then get formally the functional representation for the partition function as

$$\begin{aligned}\mathrm{Tr}\,e^{-\beta H} &= \int d\phi\,\langle\phi|e^{-\beta H}|\phi\rangle\\ &= N\int_{\text{periodic}}[d\phi]e^{i\int_{-T}^{-T-i\beta}dt\int d^3xL(\phi,d\phi/dt)}.\end{aligned}\tag{3.2}$$

Note that we have not yet specified the path of integration over time connecting the end points $-T$ and $-T-i\beta$. In principle it can be any path as long as it is within the analyticity strip and slopping downward. The so-called imaginary and the real time formalisms result from two convenient choices of the time path.

Imaginary time formalism :

It results from choosing the contour along the imaginary axis in the complex time plane from 0 to $-i\beta$ (see Figure 1a). Setting $t=-i\tau$, (3.2) becomes

$$Z = N\int[d\phi]e^{-\int_0^\beta d\tau\int d^3xL_E(\phi,d\phi/d\tau)}.\tag{3.3}$$

Being periodic in τ, $\phi(\boldsymbol{x},\tau)$ admits a fourier expansion in τ,

$$\begin{aligned}\phi(\boldsymbol{x},\tau) &= \frac{1}{\beta}\sum_n e^{i\omega_n\tau}\phi_n(\boldsymbol{x}), \qquad \omega_n = \frac{2\pi n}{\beta}\\ &= \frac{1}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_n\tau}\phi_n(\boldsymbol{k}).\end{aligned}\tag{3.4}$$

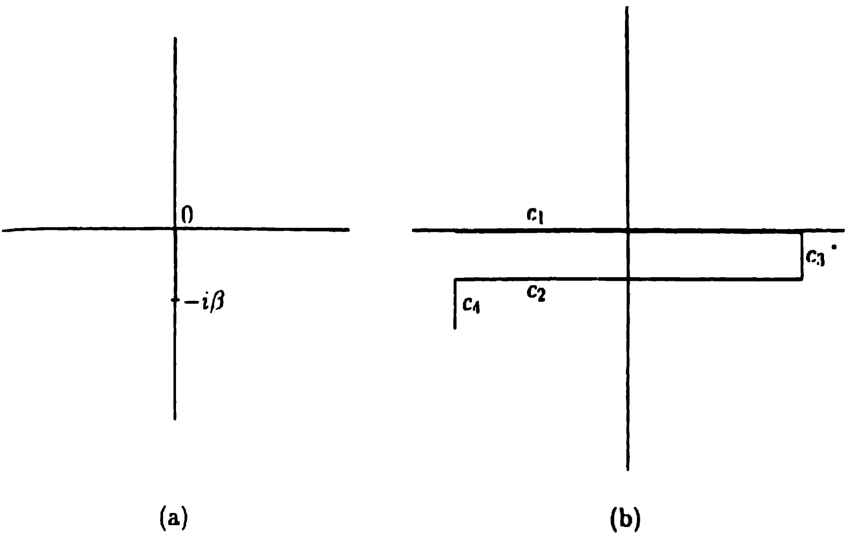


Figure 1. The time contours (a) and (b) for the imaginary and the real time formalism respectively.

Now the free action in (3.3) may be worked out to get the propagator [4]. Alternatively recall that the propagator is a periodic function in τ of period β, so that it has the Fourier series,

$$G(x-x', \tau-\tau') = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-x') + i\omega_n(\tau-\tau')}\bar{G}(k, \omega_n). \quad (3.5)$$

Noting that

$$\delta(\tau-\tau') = \frac{1}{\beta}\sum_n e^{i\omega_n(\tau-\tau')}$$

eq. (2.8) gives

$$\bar{G}(k, \omega_n) = \frac{1}{\omega_n^2 + k^2 + m^2}. \quad (3.6)$$

It is clear that the Feynman rules in the imaginary time formalism are the same as in the conventional field theory with the replacements,

$$\int \frac{d^4k}{2\pi^4} \to \frac{i}{\beta}\sum_n \frac{d^3k}{(2\pi)^3}, \quad k_0 \to i\omega_n \quad (t \to -i\tau)$$

$$i(2\pi)^4\delta^4\left(\sum k\right) \to (2\pi)^3\beta\delta_{\sum\omega_n,0}\,\delta^3\left(\sum k\right).$$

Real time formalism :

If one is interested in Green's functions with real time arguments, the imaginary time formalism is not convenient, as it has to go through a non-trivial process of analytic continuation. It is then useful to have the time integration over a path including the real axis [5]. The propagator with such a time path can be obtained by solving (2.8) subject to the KMS boundary condition (2.7), we found earlier in the operator formalism. (It can also be obtained in the path integral formalism.)

Introduce the spatial Fourier transform,

$$D_\beta(x-x') = \int \frac{d^3k}{(2\pi)^3} e^{ik(x-x')} \tilde{D}_\beta(\tau, \tau'; k),$$

where the parameter τ runs on the time contour. Now $\tilde{D}_\beta$ satisfies,

$$\left(-\frac{d^2}{dt^2} - \omega_k^2\right)\tilde{D}_\beta(\tau, \tau'; k) = \delta(\tau-\tau'), \qquad \omega_k^2 = k^2 + m^2.$$

The most general solution is obtained by adding homogeneous solutions to any particular solution, which we take to be the (zero temperature) solution with Feynman boundary condition,

$$\tilde{D}_\beta(\tau, \tau'; k) = -\frac{1}{2\omega_k}\left\{e^{-i\omega_k(\tau-\tau')}\theta_c(\tau-\tau') + e^{i\omega_k(\tau-\tau')}\theta_c(\tau'-\tau)\right.$$
$$\left. + B_1 e^{-i\omega_k(\tau-\tau')} + B_2 e^{i\omega_k(\tau-\tau')}\right\}.$$

The KMS condition (2.7) now gives

$$B_1 = B_2 = \frac{1}{e^{\beta\omega_k} - 1} \equiv B$$

getting finally

$$\tilde{D}_\beta(\tau, \tau'; k) = -\frac{i}{2\omega_k}\left(\left\{\theta_c(\tau-\tau') + B\right\}e^{-i\omega_k(\tau-\tau')}\right.$$
$$\left. + \left\{\theta_c(\tau'-\tau) + B\right\}e^{i\omega_k(\tau-\tau')}\right) \qquad (3.7)$$

A contour which gives rise to a symmetrical propagator is shown in Figure 1b. It starts at $-T$ and runs along the real axis to $+T$ (segment C_1), drops vertically from $+T$ to $+T - i\beta/2$ (segment C_3), returns parallel to the real axis to $-T - i\beta/2$ (segment C_2) and finally again drops vertically to $-T - i\beta$ (segment C_4). It can be shown that as $T \to \infty$, the generating functional factorises into a contribution from C_1 and C_2 and a contribution from C_3 and C_4. Thus for the computation of the real time Green's function, the functional integral involving C_3 and C_4 behaves like a multiplicative constant and may be dropped.

In momentum space the elements of the $2 \otimes 2$ matrix propagator can be easily obtained from (3.7)

$$D^\beta(k)_{11} = D^{\beta*}(k)_{22} = D(k) + 2\pi n(\omega_k)\delta(k^2 - m^2), \qquad (3.8)$$
$$D^\beta(k)_{12} = D^\beta(k)_{21} = 2\pi n(\omega_k)e^{\beta\omega_k/2}\delta(k^2 - m^2),$$

where $n(\omega_k)$ is the Bose distribution function, $n(\omega_k) = (e^{\beta\omega_k} - 1)^{-1}$ and $D(k)$ is the zero temperature Feynman propagator, $D(k) = i/(k^2 - m^2 + i\varepsilon)$.

Although the propagator has now a matrix structure, the topological and the combinatorial structures are the same as in the zero temperature theory. The matrix structure, arising out of the two segments C_1 and C_2 in the time path, may be interpreted as due to a doubling of the degrees of freedom : the field of type 1 living on the segment C_1 and the field of type 2 on the segment C_2, the 'thermal ghost' field. There is no direct coupling between the two types of fields and the Feynman rules for the two kinds of vertices differ by a minus sign.

Of course, we are interested only in Green's functions of type 1 fields, but the perturbation expansion brings in type 2 vertices along with the type 1 vertices. If we use only the type 1 vertices, pathological terms appear. But the contributions of type 2 vertices

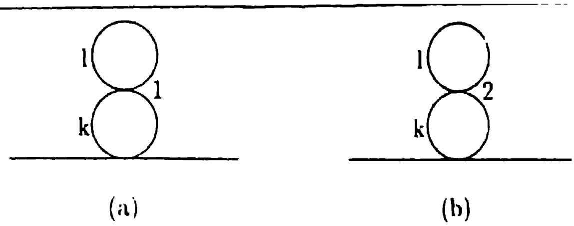


Figure 2. The double loop diagrams in the two point function for (a) the physical and (b) the ghost vertex

just cancels these terms. As an example, consider the two 2-loop diagrams of Figure 2. The contributions of these two diagrams separately are,

$$F_a = -\frac{\lambda^2}{4}\int\frac{d^4l}{2\pi^4}(D_\beta\tau)_{11}\int\frac{d^4k}{2\pi^4}(D_\beta\tau)_{11}(D_\beta\tau)_{11}$$

$$F_b = -\frac{\lambda^2}{4}\int\frac{d^4l}{2\pi^4}(D_\beta\tau)_{11}\int\frac{d^4k}{2\pi^4}(D_\beta\tau)_{12}(D_\beta\tau)_{21},$$

where the propagators are multiplied with the matrix τ, which is diagonal with elements 1 and -1. It takes into account the sign change at the type 2 (ghost) vertex. Each of the above expressions has a pathological term $\sim\left(\delta(k^2-m^2)\right)^2$. However the two terms can be added to give

$$F_a+F_b = -\frac{\lambda^2}{4}\int\frac{d^4l}{2\pi^4}(D_\beta\tau)_{11}\int\frac{d^4l}{2\pi^4}(D_\beta\tau)_{11}^2. \tag{3.9}$$

It is helpful to use the so-called mass derivative formula [6]

$$\frac{1}{n!}\left(i\frac{\partial}{\partial m^2}\right)^n D_\beta\tau = (D_\beta\tau)^{n+1} \tag{3.10}$$

to write (3.9) as

$$F_a+F_b = -\frac{i\lambda^2}{4}\int\frac{d^4l}{2\pi^4}(D_\beta\tau)_{11}\frac{\partial}{\partial m^2}\int\frac{d^4k}{2\pi^4}(D_\beta\tau)_{11}, \tag{3.11}$$

which is a well-defined expression.

Non-equilibrium in early universe :

Though a discussion of the thermal r ›n-equilibrium situation is outside the scope of this review, we nevertheless wish to point out an application of the real time formalism to a kind of non-equilibrium, which presumably took place in the early universe.

In the expanding universe there is no strict definition of thermal equilibrium. However, operationally, an equilibrium condition is reached around a time t_0, say, when the collision rate of the particles far exceeds the expansion rate of the universe. Then the density matrix is given by

$$\rho(t_0) = \frac{e^{-\beta_0 H(t_0)}}{\mathrm{Tr}\, e^{\beta_0 H(t_0)}}, \quad \beta_0 = \beta(t_0)$$

In the Heisenberg representation the density matrix is constant (even if the Hamiltonian has explicit time dependence, as is the case here). Thus the thermal average of an operator O continues to be given at later times by the expression

$$\langle O(t) \rangle = \mathrm{Tr}\, \rho(t_0) O(t) \tag{3.12}$$

even if the system ceases to be in thermal equilibrium.

Let us describe the matter in the early universe by a single real scalar field. The action for the matter field in an external gravitational field $g_{\mu\nu}$ is given by

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M_T^2\phi^2 - \frac{\lambda}{4!}\phi^4\right), \tag{3.13}$$

where the mass M_T includes the thermal contribution. In the standard cosmology the metric is taken to be homogeneous, isotropic and spatially flat,

$$ds^2 = dt^2 - a(t)^2\, d\boldsymbol{x}^2.$$

In the path integral formulation, the factor $e^{-\beta_0 H(t_0)}$ is represented as a functional integral involving the Euclidean action associated with $H\ (t_0)$. The time evolution of the field $\phi\ (\boldsymbol{x},\ t)$, on the other hand, is analysed in terms of a minkowskian path integral involving the action (3.13). To evaluate a quantity like (3.12), the two types of functional integrals need be glued together. One thus gets the time path of Figure 3, as proposed originally by Semenoff and Weiss [7].

The propagator now becomes a $3 \otimes 3$ matrix satisfying

$$K_a G_{ab}(\boldsymbol{x}, t, t') = \delta_{ab}\delta^3(\boldsymbol{x})\delta(t-t'), \tag{3.14}$$

where $$K_1 = ia^3(t)\left(\frac{d^2}{dt^2} + 3\frac{\dot a}{a}\frac{d}{dt} + \omega^2\right),$$

$$K_2 = -K_1, \quad K_3 = a_0^3\left(-\frac{d^2}{dt^2} + \omega_0^2\right),$$

and $\omega^2(t) = -a^{-2}(t)\nabla^2 + M^2(t)$, $\omega_0 = \omega(t_0)$. The boundary conditions to be imposed are obtained by matching the components of G_{ab} at the meeting points of the three segments.

The resulting propagator will have additional short distance singularities compared to the zero temperature propagator, due to the fact that the density matrix is specified sharply at $t = t_0$. They are similar to those at zero temperature in the case of a background geometry for which the derivative of the scale factor $a(t)$ changes abruptly at $t = t_0$. This additional singularity makes the field theory non-renormalizable.

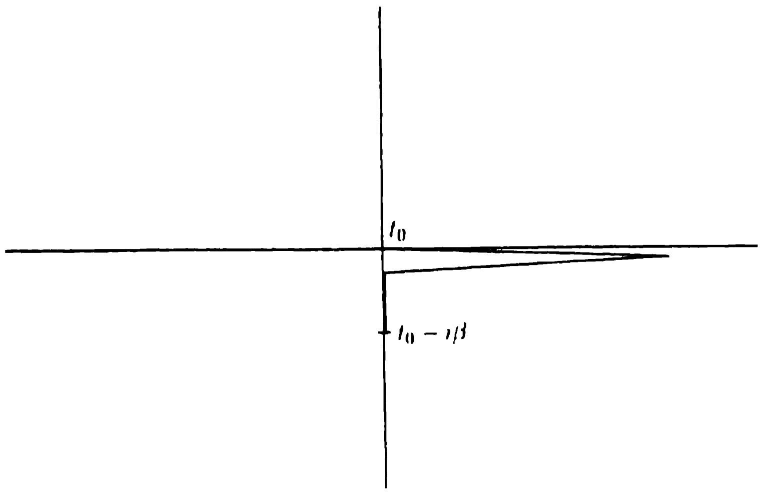


Figure 3. The time contour for problems in the early universe

This problem may be avoided [8], if we thermalise the system at a time prior to t_0 in a fictitious, static background and then follow the dynamical evolution of the Greens function as the fictitious geometry smoothly goes over to the geometry of interest. Below we first assume such a deformed geometry and then assess its effect on the propagator.

The plane wave decomposition of the propagator may be written as

$$G_{ab}(x,t,t') = N_a(t)N_b(t')\int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x}\bar{G}_{ab}(k,t,t'), \tag{3.15}$$

where $$N_a(t) = \begin{cases} [2\pi a(t)]^{-3/2}, & a = 1, 2 \\ [2\pi a_0]^{-3/2}, & a = 3 \end{cases}.$$

We first write the Minkowski space mode function,

$$\left(\frac{d^2}{dt^2} + M_T^2(t) + \frac{k^2}{a^2} - \frac{3}{4}\left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right)\right)f^{\pm}(t) = 0 \tag{3.16}$$

with the boundary conditions $f^{\pm}(t_0)=1, \dot{f}^{\pm}(t_0)=\mp i\omega_0$. They are then extended to functions defined on the complex contour by

$$f_a^{\pm}(t) = \begin{cases} f^{\pm}(t), & a=1,2 \\ e^{\pm\omega_0 t}, & a=3 \end{cases}$$

It is now simple to write the propagator on the complex contour as

$$\begin{aligned} \bar{G}(\boldsymbol{k},\tau,\tau') = & \left\{\theta_c(\tau-\tau')+B\right\}f^{+}(\tau)f^{-}(\tau') \\ & + \left\{\theta_c(\tau'-\tau)+B\right\}f^{-}(\tau)f^{+}(\tau') \end{aligned} \tag{3.17}$$

with $B=(e^{\beta_0\omega_0}-1)^{-1}$. Note the remarkable similarity of this propagator with the flat-space propagator in (3.7). The density function refers only to the time t_0. It can now be easily cast in the form of a $3\otimes 3$ matrix.

Coming back to the question of using the deformed scale factor, we note that it cannot affect the mode functions significantly as long as

$$M^2 \gg \frac{\dot{a}^2}{a^2}, \frac{\ddot{a}}{a}, \quad \text{at } t \simeq t_0, \tag{3.18}$$

which for $M \sim gT$, gives

$$gT \gg \frac{\dot{a}}{a}. \tag{3.19}$$

This condition should be compared with the condition for maintaining the thermal equilibrium in the expanding universe. The collision rate $\sim g^4 T$, while the expansion rate $\sim \dot{a}/a$. Thus thermal equilibrium requires,

$$g^4 T \gg \frac{\dot{a}}{a}. \tag{3.20}$$

Thus once condition (3.20) is satisfied, (3.19) is automatically satisfied.

The reader may recall the "infrared problem", encountered in setting up quantum field theory on curved space-time. If the momentum k/a and $M_T(t)$ are so small that the curvature term dominates in eq. (3.16) for the mode function, we cannot define them to belong to positive and negative frequencies. The resulting field theory appears ambiguous.

The present formulation of the quantum field theory in the cosmological context avoids this ambiguity. It is, of course, essential that the evolution passes through a phase where the condition (3.18) holds, *i.e.*, the effective mass is large enough compared to the expansion rate. It ensures the existence of positive and negative fequency mode functions around the time t_0. Later on, the scale factor and the mass may well develop in such a way that the expansion rate exceeds the mass. But once the system is in a thermal state at t_0, its evolution can be traced on the basis of the thermal propagator, irrespective of whether or not the condition (3.18) continues to hold.

4. Resummation at high temperature

At high enough temperature the QCD medium dissociates into quarks and gluons with simultaneous weakening of the strong interaction, so that the ordinary perturbation expansion is expected to hold. However, this expectation is naive : Loop corrections tend to be as large as the tree level contributions at high temperature. Indeed, it is this breakdown of perturbation expansion which constitutes the earliest example of application of thermal field theory to particle physics, *viz*, the restoration of symmetry at finite temperature [9–11]. The gauge symmetry breaking at zero temperature by the Higgs potential is restored at high temperature when the tree level (tachyonic) mass of the Higgs field is compensated by its loop correction.

This situation in thermal field theory calls for a resummation and a consequent reformulation of the perturbative expansion. Although loop corrections were included earlier in the propagators by several authors [12], the systematic approach to the problem is due to Braaten and Pisarski [13].

Consider a field theory at high temperature, when the bare masses of the particles are negligible compared to the temperature. Then one has natural momentum scales T, gT and so on, where g is the (small) coupling constant. Restricting to one loop diagrams, we have two scales to consider : the hard, of order T and the soft, of order gT. A momentum is called hard, if the magnitude of any of its components is of order T; it is called soft, if the magnitudes of all its components are of order gT. The resummation is needed only for amplitudes with soft external momenta. One finds that the contribution of one-loop diagrams to such amplitudes, which are of the order of the tree graphs, arise from hard internal momenta. The dominance of these so-called hard thermal loops (HTL) is understood by recalling that the temperature in the density distribution provides the cut-off for the otherwise divergent momentum integrals. In addition, the Landau singularities also contribute to the enhancements.

Although the non-abelian gauge theories present all the aspects of the resummation programme, the basic ideas can be illustrated by a consideration of the scalar field theory [14], to which we now turn.

Scalar field theory :

Consider the field theory of a single massless scalar field,

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{g^2}{4!}\phi^4 \tag{4.1}$$

Let us examine the one-loop contributions to different n-point functions. The self energy to one loop is given by the tadpole diagram, Figure 4a. Subtracting off the zero temperature part, it is given by

$$\delta\Sigma = \frac{1}{2}g^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}n(k) = \frac{1}{24}g^2T^2. \tag{4.2}$$

Summing over the series of one particle reducible tadpole diagrams, we get the effective propagator to one loop,

$$^{*}D(k) = \frac{1}{k^2 + \delta\Sigma} \tag{4.3}$$

Thus if the momentum k_μ is hard, the effective propagator is, to a good approximation, the same as the bare one. On the other hand, if k_μ is soft, the loop correction is as large as the bare inverse propagator.

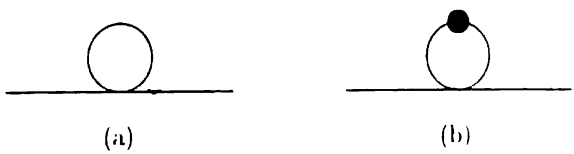


Figure 4. The two point function with (a) the bare propagator and (b) the effective propagator

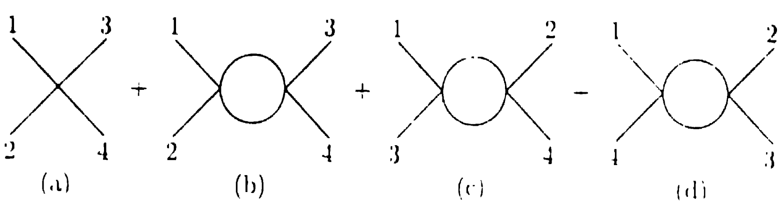


Figure 5. The four point in (a) tree level and (b), (c) and (d) at one loop

Next consider the diagrams for the 4-point function in Figure 5. The bare vertex is g^2. The contribution of the other three diagrams are similar. To illustrate the nature of the contribution expected, consider Figure 5b. Let $p = p_1 + p_2 = p_3 + p_4$. Then the diagram contributes as

$$\Gamma_4(p) = g^4 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_1 2E_2} \left[(1 + n_1 + n_2) \left(\frac{1}{p_0 - E_1 - E_2} + \frac{1}{p_0 + E_1 + E_2} \right) \right.$$

$$\left. -(n_1 - n_2) \left(\frac{1}{p_0 - E_1 + E_2} + \frac{1}{p_0 + E_1 - E_2} \right) \right]. \tag{4.4}$$

where $E_1 = |\boldsymbol{k}|$, $E_2 = |\boldsymbol{p} - \boldsymbol{k}|$ and n_1 and n_2 are the corresponding Bose densities. The retarded amplitude is given by replacing p_0 by $p_0 + i\varepsilon$. Consider now $p_0 > 0$. At $T = 0$, the absorptive part (given by 1 in the factor $(1 + n_1 + n_2)$) corresponds to both the internal lines having positive energies. But for $T \neq 0$, there arise additional contributions corresponding to one of the lines having positive and the other negative energy. It represents Landau damping, where one particle is absorbed from the heat bath and the other emitted into it. The Landau terms have discontinuities below the light cone, while the non-Landau terms give rise to the usual discontinuity above the physical threshold.

We now evaluate the contribution of (4.4) for soft external momentum p_μ. The $T = 0$ contribution, after renormalization, is proportional to $g^4 \ln (p^2/\mu^2)$, where μ is the renormalisation scale. It is $\sim (g^2 \ln g)$ compared to the bare vertex.

The other terms in (4.4) involve the density function $n(k)$. It is easy to estimate these terms for soft loop momentum $\boldsymbol{k}$, for which $n(k) \sim T/k$. With the internal and external momenta both soft, the only scale is gT, so that the integral is $g^4 n(k) \sim g^3$. It is thus suppressed by a power of g compared to the bare vertex.

The remaining domain of integration is over the hard internal momenta. When p is soft and k hard,

$$E_1 = |\boldsymbol{k}|, \qquad E_2 = |\boldsymbol{p}-\boldsymbol{k}| \simeq |\boldsymbol{k}| - |\boldsymbol{p}|\cos\theta,$$

$$p_0 \pm (E_1 + E_2) \simeq \pm 2|\boldsymbol{k}|, \qquad p_0 \pm (E_1 - E_2) \simeq p_0 \pm |\boldsymbol{p}|\cos\theta,$$

$$n_1 \simeq n_2 \simeq 1, \qquad n_1 - n_2 \simeq -\frac{|\boldsymbol{p}|\cos\theta}{T} n_1 (1 + n_1).$$

Then the Landau terms contribute as g^4, while the non-Landau terms given $\sim g^4 \ln (T/p)$, where p is the soft momentum. So the contribution of the HTL is also $\sim g^2 \ln g$ compared to the bare vertex.

It is important here to notice that though the Landau terms have energy denominators larger by a factor of $1/g$ with respect to the those in non-Landau terms, the former could not dominate because of $(n_1 - n_2) \sim p/T$. The situation will be different for the QCD theory, where the fermion density function is ~ 1 for hard internal momenta.

Similar analysis shows that the corrections to all higher point vertex functions are small compared to the their tree level contributions. Thus in the scalar theory the only HTL is in the two point function.

Having obtained the effective propagator, we can now construct the effective perturbation expansion to replace the naive one. Rewrite the Lagrangian (4.1) as

$$L = L_0 + \delta L,$$

where

$$L_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_T^2 \phi^2 - \frac{g^2}{4!}\phi^4,$$

$$\delta L = \frac{1}{2} m_T^2 \phi^2,$$

$$m_T^2 = \frac{1}{24} g^2 T^2. \tag{4.5}$$

The effective expansion is obtained by constructing the usual perturbation expansion with L_0, which is the same as the earlier bare one except for the thermal mass term in the propagator. The counterterm δL is a reminder to avoid double counting, *i.e.*, to exclude the contribution of the hard internal momenta in loops appearing in the propagator with soft momenta.

As an example of the effective perturbation expansion, we calculate the leading correction to the effective self-energy to one loop. It is just the same tadpole diagram with the bare propagator replaced by the effective propagator (Figure 4b),

$$\delta\Sigma + {}^*\Sigma = \frac{1}{2}g^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E}\,2n(E),,$$

$$E = \sqrt{k^2 + m_T^2} \tag{4.6}$$

where again the zero temperature contribution has been subtracted off. For small values of m/T, it can be evaluated as

$$\delta\Sigma + {}^*\Sigma = m_T^2\left(1 - \frac{3m_T}{\pi T} + \cdots\right). \tag{4.7}$$

Since the counterterm subtracts off the hard thermal loop ($\equiv \delta\Sigma$), we are left with soft internal momenta $k \sim gT$ in eq. (4.6), for which $n(k) \sim T/k$ as we noticed already. Over such momenta it is of the order of

$$g^2 \int_{\rm soft} \frac{d^3k}{k}\frac{T}{k} \sim g^3T^2,$$

which is the origin of the second term in (4.7). (In the integration region over the hard internal momenta, (4.6) has also a correction along with HTL. But it is $\sim g^2$ with respect to the latter.)

Hot QCD theory :

The existence of hard thermal loops in QCD theory may be investigated essentially in the same way as we did for the scalar field theory. Here the source of complication lies in the fact that, unlike the case for the scalar theory where HTL exists only in the two-point function, all N-point functions of gluons and all $(N-2)$-point function of gluons and a quark pair have hard thermal loops.

Let us illustrate the resummation programme for QCD by discussing the gluon damping rate in a schematic way [15]. Dropping the colour and the space-time indices, the bare gluon propagator and the bare three- and the four-gluon vertices are written respectively as $\Delta(p)$, $g\Gamma_3$ and $g^2\Gamma_4$, where the momentum dependence of the vertices are also omitted. The corresponding effective quantities are written as ${}^*\Delta$, $g{}^*\Gamma_3$ and $g^2{}^*\Gamma_4$,

$${}^*\Delta(P) = \frac{1}{p^2 + \delta\Pi}, \quad {}^*\Gamma_n = \Gamma_n + \delta\Gamma_n, \quad n = 3, 4 \tag{4.8}$$

where $\delta\Pi$ and $\delta\Gamma_n$ are the contributions of the hard thermal loops in the gluon self-energy and the vertices respectively.

The effective expansion for the gluon self-energy to one loop can be written schematically as

$$ {}^*\Pi(p) = g^2 \int d^4k.{}^*\Gamma_3.{}^*\Delta.{}^*\Gamma_3.{}^*\Delta + g^2 \int d^4k.{}^*\Gamma_4.{}^*\Delta $$

$$ + \text{contributions of the counterterms}, \tag{4.9} $$

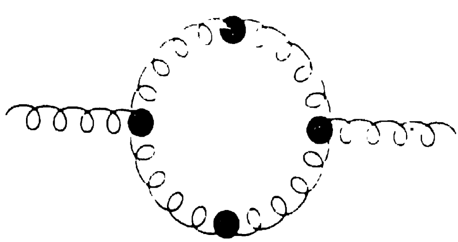
where the first two terms correspond to the diagrams of Figure 6. The zero of the inverse gluon propagator is given by

$$ \omega^2 - p^2 - \delta\Pi - {}^*\Pi(E, p) = 0, \quad \delta\Pi(E, 0) \equiv m_g \sim gT, \tag{4.10} $$

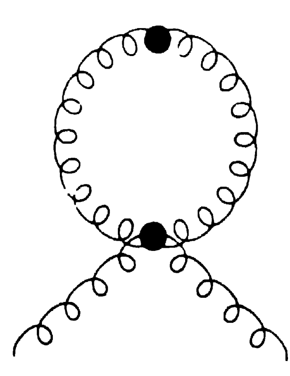
so that the gluon damping rate at zero momentum is given to lowest order in g by

$$ \gamma(0) = \frac{1}{2m_g} \cdot \mathrm{Im}\, {}^*\Pi(E, 0). \tag{4.11} $$

We now estimate the order of magnitude of the contributions to ${}^*\Pi$ coming from the soft and the hard internal momentum regions. In the following we denote a loop corection as $O(g^n)$, if it is of order g^n with respect to the corresponding tree amplitude.



(a)



(b)

Figure 6. Effective expansion of the gluon self-energy to one loop

Let us first show that the integration over hard momentum in (4.9) does not give any contribution to $\gamma(0)$. It gives $O(1)$ contribution but the resummation programme is just designed to cancel it with counterterms. But this region also give terms $O(g)$, $O(g^2)$, ..., which were neglected in arriving at the HTL contribution. However they cannot contribute to the discontinuity in (4.11) for kinematical reasons. When the internal lines are put on mass shell, both lines will be hard. So the discontinuities are either of the Landau damping type in the unphysical region or far above the threshold.

We are then left with soft loop momenta. With internal and external momenta both soft, the effective propagators and the vertices are of the same order as the bare ones. Also the only mass scale in the integral is gT. Thus if the outside factors of g and the density functions are removed, the remaining integral $\sim(gT)^2$. Hence for soft momenta, the integral must be $\sim g^2 n(E)\ (gT)^2 \sim g^3T^2$. Thus with $m_g \sim gT$, we get from eq. (4.11), $\gamma(0) \sim g^2T$. Nontrivial loop calculations involving effective propagators are required to find the constant of proportionality [15].

5. Chiral perturbation theory at low temperature

Chiral perturbation theory (χPT) [16], so successful in analysing the low energy structure of the QCD theory, can naturally be extended to finite temperature to describe the thermodynamic and other thermal properties of the theory.

Let us first briefly recall the basis of χPT. Consider only the u and the d quarks. To a good approximation, they may be taken to be massless. Then the QCD Lagrangian becomes,

$$\mathcal{L}_{\rm QCD} = \bar{q}(x)\gamma^{\mu}\left(\partial_{\mu} - iA_{\mu}(x)\right)q(x), \tag{5.1}$$

where $q(x)$ has two components in flavour space and $A_\mu(x)$ is the colour $SU(3)$-matrix valued gauge potential. It is invariant under independent isospin transformations of the left-handed and the right-handed quark fields,

$$q_R \xrightarrow{g} V_R q_R, \quad q_L \xrightarrow{g} V_L q_L, \quad g \in SU(2)_L \otimes SU(2)_R. \tag{5.2}$$

It is generally believed that this chiral symmetry is spontaneously broken down to $SU(2)_V$ by the ground state of the theory having a quark condensate. It gives rise to 3 massless, pseudoscalar Goldstone bosons, to be identified with the pion triplet. This symmetry is again broken explicitly by the quark mass term, which gives pion its mass.

It can be shown that the above transformation law for the quarks induce the transformation law,

$$U(x) \rightarrow V_R U(x) V_L^{-1}, \tag{5.3}$$

on the matrix $U(x)$ of the pion fields ϕ, $U(x) = e^{i\phi\cdot\tau/F}$, where F can be identified with the value of the pion decay constant in the chiral limit. Thus although $U(x)$ transforms linearly, $\phi(x)$ transforms non-linearly.

It is now easy to construct the effective Lagrangian invariant under the transformation law (5.3). There cannot be any term without derivatives. The pieces in the Lagrangian can be ordered according to the number of derivatives,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots. \tag{5.4}$$

The lowest order term is the one with two derivatives,

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\,\mathrm{Tr}(\partial_\mu U \partial^\mu U). \tag{5.5}$$

Chiral symmetry is broken by terms in $\mathcal{L}_{\text{eff}}$ which contain the (diagonal) mass martix of u and d quarks. To lowest order

$$\mathcal{L}_M = \frac{1}{2} F^2 B\,\mathrm{tr}\left[m(U + U^{\dagger})\right], \tag{5.6}$$

where B is given by $M_\pi^2 = (m_u + m_d)B$. By χPT one refers to the effective field theory constructed with this effective Lagrangian, which combines the expansion in powers of momenta with expansion in powers of m_u and m_d.

As already pointed out in Sec. 3, the partition function can be converted to a path integral formula, leaving the time contour free to choose. To compute the static thermodynamic properties it is convenient to use the imaginary time formalism. With the effective chiral Lagrangian (5.4–6), we get

$$Z = \int [dU]\, e^{\int_0^\beta dx_4 \int d^3x \mathcal{L}_{\text{eff}}}, \tag{5.7}$$

the path integration extending over all pion field configurations which are periodic in the euclidean time direction, $U(\boldsymbol{x}, x_4 + \beta) = U(\boldsymbol{x}, x_4)$.

The partition function has been evaluated to 3 loops in χPT by Gerber and Leutwyler [17]. They carry out the entire calculation in configuration space, where the pion propagator can be written as

$$G() = \sum_{n=-\infty}^{\infty} \Delta(\boldsymbol{x}, x_4 + n\beta),$$

where $\Delta(x)$ is the euclidean propagator at zero temperature.

For massless quarks the pressure has been calculated to give

$$P = \frac{1}{30}\pi^2 T^4 \left(1 + \frac{T^4}{36F^4}\ln\left(\Lambda_p/T\right) + O\left(T^6\right)\right), \tag{5.8}$$

where $\Lambda_p \simeq 275$ MeV. The leading (one-loop) contribution is the familiar Bose gas term. The two-loop contribution is zero, as the $\pi\pi$ scattering amplitude vanishes in the forward direction, on account of the Adler zero. We refer to the original papers [17] for the details of the calculation. Below we only discuss the results for the quark and the gluon condensates.

To get the quark condensate we have to perturb the chiral Hamiltonian H^0_{QCD} by the quark mass term

$$Z = \mathrm{Tr}\, e^{-\beta\left(H^0_{QCD} + \int d^3x m\bar{q}q\right)}, \tag{5.9}$$

where m is the quark mass. We get

$$\langle\bar{q}q\rangle = Z^{-1}\mathrm{Tr}\, e^{-\beta H}\bar{q}q = -\frac{1}{\beta V}\frac{\partial \ln Z}{\partial m}, \tag{5.10}$$

where the volume V goes to infinity at the end. Since $\ln Z = -\beta V(\varepsilon_0 - P)$, where ε_0 and P are the vacuum energy density and the pressure respectively, we get

$$\langle\bar{q}q\rangle = \frac{\partial\varepsilon_0}{\partial m} - \frac{\partial P}{\partial m} = \langle 0|\bar{q}q|0\rangle - \frac{\partial P}{\partial m}. \tag{5.11}$$

Thus it is necessary to work out the pressure up to the term linear in the quark mass. One obtains finally [17]

$$\langle qq\rangle = \langle 0|qq|0\rangle\left(1 - x - \frac{1}{6}x^2 - \frac{16}{9}x^3\ln\left(\Lambda_q/T\right) + \cdots\right),$$
$$x = T^2/8F^2, \tag{5.12}$$

where $\Lambda_q = 470$ MeV.

The temperature dependence of the gluon condensate can also be determined [18]. The trace anomaly reads as

$$\Theta^\mu_\mu = \frac{\beta(g)}{2g^3}g^2G^a_{\mu\nu}G^{\mu\nu a} \equiv -G^2, \tag{5.13}$$

where $\beta(g)$ denotes the beta-function of the QCD theory, $\beta(g) = -\frac{g^3}{(4\pi)^2}(11 - \frac{2}{3}n_f)$, n_f being the number of quark flavours. To normalise Θ^μ_μ such that it is zero in vacuum, we write

$$\Theta^\mu_\mu = -G^2 + \langle 0|G^2|0\rangle$$

giving
$$\langle G^2\rangle = \langle 0|G^2|0\rangle - \langle\Theta^\mu_\mu\rangle. \tag{5.14}$$

The thermal average of Θ^μ_μ can be related to pressure

$$\langle\Theta^\mu_\mu\rangle = \varepsilon - 3P = T^5\frac{\partial}{\partial T}\left(\frac{P}{T^4}\right),$$

on using the thermodynamic relation

$$\varepsilon = T\frac{\partial P}{\partial T} - P.$$

Inserting the chiral perturbation theory value for P, we get

$$\left\langle \Theta^{\mu}_{\mu} \right\rangle = \frac{\pi^2}{270}\frac{T^8}{F^4}\left(\ln\left(\Lambda_p/T\right) - \frac{1}{4}\right) + \cdots \tag{5.15}$$

The series (5.12) and (5.15) for the two condensates in powers of (T/F) must be treated as asymptotic [19]. Any non-Goldstone particle of mass M gives a contribution of $\sim e^{-m/T}$, which does not show up at any finite order. Though negligible below $T \doteq 140$ MeV, such contributions grow rapidly with further increase of temperature.

Corrections from non-zero quark masses and contributions from more massive states have been included in the formula for the quark condensate. Even then its validity is expected up to $T \sim 150$ MeV, as beyond this temperature the interaction of massive states with the pions and among themselves become significant. However since the condensate falls off rapidly at the upper end of this range, it is meaningful to make an estimate for the critical temperature from the corrected formula, which gives $T_c = 190$ MeV.

Although the gluon condensate also melts with growing temperature, the melting takes place much more slowly than in the case of the quark condensate. The difference may be traced to the fact that while $\bar{q}q$ transforms in a nontrivial manner under chiral transformations, Θ^{μ}_{μ} and $G^{a}_{\mu\nu}G^{\mu\nu a}$ are chiral singlets. The gluon condensate is a parameter associated with non-perturbative scale breaking effects and does not represent an order parameter.

6. Conclusion

We began by reviewing the path integral formulation of the perturbation theory at finite temperature. The discussion concerned only the thermal equilibrium, except for the special non-equilibrium situation which is relevant in the context of the early universe. Then we explained, in the simpler context of ϕ^4 theory, the necessity of resummation of the perturbation series to restore its validity at high temperature. As an example in QCD theory, we sketched the calculation of the gluon damping rate.

We must add that the subject of resummation at high temperature is far from being closed. As already pointed out by Braaten and Pisarski [13], there appear collinear divergences when the external particles are on the mass shell or massless. This fact and the absence of magnetic mass generation in perturbation theory give rise to a number of problems, which are actively persued at present.

At low temperature in the hadronic phase, the QCD theory as such is very complicated and is replaced by its symmetries as embodied in χPT. Here we discussed mainly the results it gives for the quark and the gluon condensates. Among the numerous applications of χPT at finite temperature, we mention the calculation of the effective masses of hadrons [20].

Not covered in this review is the method of QCD sum rules at finite temperature [21]. Is has the potential to provide substantial information on the thermal properties of QCD theory. Unfortunately all the works done so far with these sum rules are incomplete and hence unreliable in that not all the operators of leading dimension, which appears in the operator product expansion of the two point functions [22], are included in the sum rules.

Clearly none of the methods are adequate to analyse the intermediate region of temperature, where the QCD medium is supposed to undergo a phase transition. The appropriate method here is the numerial analysis on the lattice, which again is not discussed in this review.

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